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TRANSFORMATION OF LONG NONLINEAR WAVES IN A TWO-LAYER VISCOUS FLUID BETWEEN A GENTLY SLOPING LID AND BOTTOM

G. A. Khabakhpashev

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Dynamics of three-dimensional disturbances of the interface between two fluid layers of different densities is considered analytically and numerically. An evolutionary integrodifferential equation is derived, which takes into account long-wave contributions of inertia of the layers and surface tension, small but finite amplitude of disturbances of the interface between two incompressible immiscible fluids, gentle slopes of the lid and bottom, and nonstationary shear stresses at all boundaries. Numerical solutions of this model equation for several (most typical) nonlinear problems of transformation of two- and three-dimensional waves are obtained.

Key words: *viscous fluid, interface, long waves, nonlinear disturbance.*

Introduction. Despite the long story of investigations of finite-amplitude waves, almost all stringent theoretical results were obtained under the assumption of an ideal fluid (see the monographs [1–6] and the references therein). Therefore, approximate models are used in studying the evolution of even two-dimensional, weakly nonlinear, solitary disturbances of the interface between two layers of a viscous medium between a horizontal lid and a horizontal bottom (see, e.g., [7–9]).

Thus, Koop and Butler [7] made two attempts to take into account dissipation of internal waves with the use of approaches developed for a shallow homogeneous fluid with a free surface [10, 11]. The first approach implied the use of energy relations and the expression for the shear stress averaged over the wave length. This made it possible to derive a formula for decay of solitary disturbances. In the second attempt, by means of a standard decomposition with respect to a small parameter and the Fourier transform for deviation of the free boundary, an equation of the Korteweg–de Vries type with a dissipative integral term was derived, which allowed calculation of the transformation of waves during their propagation. In addition, it was assumed in both cases that the values of friction in the upper and lower layers at the interface were equal to shear stresses at the lid and at the bottom, respectively. A comparison of results calculated by the model developed in [7] with the experimental data obtained in [12], however, revealed a significant difference.

Actually, Leone et al. [8] studied dissipation of solitary disturbances in a two-layer fluid with a free surface, but the solution could be readily adopted to the problem with a solid lid [12]. In contrast to the first approach [7], dissipation at the interface was taken into account more correctly in [8]. Nevertheless, a comparison with the experiments [12] confirmed the conclusion that the use of energy relations and the shear stress averaged over the wave length does not yield good agreement between the theory and the experiment [9]. Note that a similar situation is observed in the presence of a free surface as well [13].

Khabakhpashev [9] derived an evolutionary equation of the Boussinesq type with an additional convolution-containing term and demonstrated that this model offers an adequate description of the profile of experimentally observed waves. The objective of the present work is to extend the latter model to the case of essentially three-dimensional disturbances in a two-layer medium with a gently sloping lid and bottom.

Kutateladze Institute of Thermophysics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090; theory@itp.nsc.ru. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 46, No. 6, pp. 45–57, November–December, 2005. Original article submitted January 14, 2005.

Formulation of the Problem and Simplification of Initial Equations. The initial equations of continuity and the Stokes equations for an incompressible fluid can be written in the following form:

$$\boldsymbol{\nabla}\cdot\boldsymbol{u}_l+\frac{\partial w_l}{\partial z}=0; \tag{1}$$

$$\frac{\partial \boldsymbol{u}_l}{\partial t}+\boldsymbol{u}_l\cdot\boldsymbol{\nabla}\boldsymbol{u}_l+w_l\,\frac{\partial \boldsymbol{u}_l}{\partial z}+\frac{1}{\rho_l}\,\boldsymbol{\nabla}p_l=\nu_l\Big(\nabla^2\boldsymbol{u}_l+\frac{\partial^2\boldsymbol{u}_l}{\partial z^2}\Big); \tag{2}$$

$$\frac{\partial w_l}{\partial t}+\boldsymbol{u}_l\cdot\boldsymbol{\nabla}w_l+w_l\,\frac{\partial w_l}{\partial z}+\frac{1}{\rho_l}\,\frac{\partial p_l}{\partial z}+g=\nu_l\Big(\nabla^2 w_l+\frac{\partial^2 w_l}{\partial z^2}\Big). \tag{3}$$

Here $\boldsymbol{u}$ and w are the horizontal and vertical components of the fluid velocity vector, the operator $\boldsymbol{\nabla}$ is determined in the horizontal plane, z is the vertical coordinate, t is the time, ρ is the fluid density, p is the pressure in the layer, ν is the kinematic viscosity of the fluid, and g is the acceleration of gravity; the subscript l equals 1 and 2 for the upper and lower layers, respectively.

We introduce the following assumptions: the stationary components of the flows of immiscible fluids equal zero; the "wave length" λ is significantly greater and the disturbance amplitude η_a is significantly smaller than the equilibrium depths of the layers h_l ($h_l/\lambda\sim\varepsilon^{1/2}$ and $\eta_a/h_l\sim\varepsilon$, where ε is a small parameter); the capillary effects are low [the modified Bond number is $\mathrm{Bo}=(\rho_2-\rho_1)\,g\,h_1h_2/\sigma>1$, where σ is the surface tension]; the motionless solid lid and bottom are gently sloping ($|\boldsymbol{\nabla}h_l|\sim\varepsilon^{3/2}$); the arising boundary layers remain thin, i.e., the time needed for the boundary layers to grow over the entire thickness of the fluid is much greater than the characteristic time of wave passage through an arbitrary point of the examined domain t_w (the number of hydrodynamic homochronicity is $\mathrm{Ho}_{\nu l}=\nu_l t_w/h_l^2\sim\varepsilon^2$). Hence, the resultant flow is potential ($\partial\boldsymbol{u}_l/\partial z=\boldsymbol{\nabla}w_l$) everywhere except for narrow boundary regions. The nonlinear terms in Eqs. (3) can be omitted as terms of negligible orders of smallness ($\boldsymbol{u}_l\cdot\boldsymbol{\nabla}w_l/g\sim\varepsilon^2$ and $w_l^2/u_l^2\sim\varepsilon$). Moreover, owing to the assumptions made, it is possible to neglect the first terms in the right sides of Eqs. (2) and all right sides in Eqs. (3). Thus, dissipation is negligible in considering the vertical components of equations of momentum balance. As a result, we obtain a simplified system of equations of motion

$$\frac{\partial \boldsymbol{u}_l}{\partial t}+\boldsymbol{\nabla}\Big(\frac{u_l^2}{2}+\frac{p_l}{\rho_l}\Big)+w_l\,\frac{\partial \boldsymbol{u}_l}{\partial z}=\nu_l\,\frac{\partial^2\boldsymbol{u}_l}{\partial z^2}; \tag{4}$$

$$\frac{\partial w_l}{\partial t}+\frac{1}{\rho_l}\,\frac{\partial p_l}{\partial z}+g=0. \tag{5}$$

We accept the traditional boundary conditions: no-slip condition at the lid and bottom, continuity of all components of velocity vectors of the fluid and shear stresses at the interface, and kinematic conditions and a condition of a capillary jump of pressure at this surface:

$$\boldsymbol{u}_l=w_l=0\qquad\text{for}\quad z=-(-1)^l h_l,$$

$$\boldsymbol{u}_1=\boldsymbol{u}_2=\boldsymbol{u}_i,\qquad w_1=w_2=w_i=\frac{\partial\eta}{\partial t}+\boldsymbol{u}_i\cdot\boldsymbol{\nabla}\eta,$$

$$\nu_1\rho_1\,\frac{\partial\boldsymbol{u}_1}{\partial z}=\nu_2\rho_2\,\frac{\partial\boldsymbol{u}_2}{\partial z}=\boldsymbol{\tau}_i,\qquad p_{1i}=p_{2i}+\sigma\nabla^2\eta\qquad\text{for}\quad z=\eta(t,x,y).$$

Here η is the disturbance of the interface, $\boldsymbol{\tau}_i$ is the shear stress at this surface, and p_{li} are the values of pressure at this interface. A schematic of the wave process examined is shown in Fig. 1.

Integrating Eqs. (5) with respect to the coordinate z from z to η and using the dynamic boundary condition on the free surface, we find the pressure profiles in each fluid:

$$\frac{p_l}{\rho_l}=\frac{p_{li}}{\rho_l}+g\,(\eta-z)+\int\limits_z^\eta\frac{\partial w_l}{\partial t}\,dz.$$

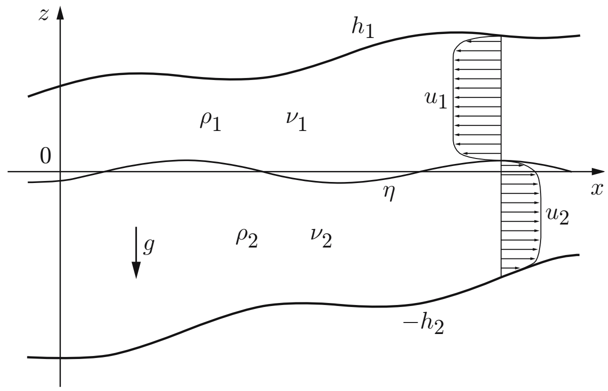

Fig. 1. Schematic of the wave process in a shallow two-layer fluid with a gently sloping lid and bottom.

Substituting these expressions into Eqs. (4), we rewrite the latter in the form

$$\frac{\partial \boldsymbol{u}_l}{\partial t} + \boldsymbol{\nabla}\Big(g\eta + \frac{u_l^2}{2} + \frac{p_{li}}{\rho_l} + \int\limits_z^0 \frac{\partial w_l}{\partial t}\,dz\Big) + w_l\,\frac{\partial \boldsymbol{u}_l}{\partial z} = \nu_l\,\frac{\partial^2 \boldsymbol{u}_l}{\partial z^2}. \tag{6}$$

The upper limit of the integral is replaced because we neglect quantities of the third and higher orders of smallness in the approximation considered (long waves of moderate amplitude). In addition, in a fluid with thin boundary layers, we can use simple profiles of normal components of fluid velocities:

$$w_1 = \Big(1 - \frac{z}{h_1}\Big)\frac{\partial\eta}{\partial t}, \qquad w_2 = \Big(1 + \frac{z}{h_2}\Big)\frac{\partial\eta}{\partial t}.$$

The omitted corrections are again characterized by the next (negligible) orders of smallness. We substitute these formulas into the equations of horizontal motion (6) to obtain

$$\frac{\partial \boldsymbol{u}_l}{\partial t} + \boldsymbol{\nabla}\Big[g\eta + \frac{p_{li}}{\rho_l} + \frac{u_l^2}{2} - z\Big(1 + (-1)^l\,\frac{z}{2h_l}\Big)\frac{\partial^2\eta}{\partial t^2}\Big] + w_l\,\frac{\partial \boldsymbol{u}_l}{\partial z} = \nu_l\,\frac{\partial^2 \boldsymbol{u}_l}{\partial z^2}. \tag{7}$$

Then we integrate Eqs. (1) and (7) with respect to z from $-h_2$ to η for $l = 2$ and from η to h_1 for $l = 1$ and introduce the values of fluid velocities averaged over the layer depths $\langle \boldsymbol{u}_l\rangle$ to obtain

$$\frac{\partial\eta}{\partial t} + \boldsymbol{\nabla}\cdot[(-1)^l h_l + \eta]\langle \boldsymbol{u}_l\rangle = 0; \tag{8}$$

$$\frac{\partial([h_l + (-1)^l\eta]\langle \boldsymbol{u}_l\rangle)}{\partial t} + [h_l + (-1)^l\eta]\,\boldsymbol{\nabla}\Big(g\eta + \frac{p_{li}}{\rho_l}\Big)$$

$$+ h_l\boldsymbol{\nabla}\Big(\langle u_l\rangle^2 + (-1)^l\,\frac{h_l}{3}\,\frac{\partial^2\eta}{\partial t^2}\Big) = \frac{(-1)^l}{\rho_l}\,(\boldsymbol{\tau}_i - \boldsymbol{\tau}_l), \tag{9}$$

where $\boldsymbol{\tau}_l = \nu_l\rho_l\,\partial\boldsymbol{u}_l/\partial z$ for $z = -(-1)^l h_l$, and integration by parts, no-slip conditions for the fluid at both solid boundaries, and continuity equations are used:

$$\int\limits_\eta^{h_1} w_1\,\frac{\partial \boldsymbol{u}_1}{\partial z}\,dz = \frac{\partial\eta}{\partial t}\,\boldsymbol{u}_i - \int\limits_\eta^{h_1} \boldsymbol{u}_1\,\frac{\partial w_1}{\partial z}\,dz = \frac{\partial\eta}{\partial t}\,\boldsymbol{u}_i + \int\limits_\eta^{h_1} \boldsymbol{u}_1\cdot\boldsymbol{\nabla}\boldsymbol{u}_1\,dz = \frac{\partial\eta}{\partial t}\,\boldsymbol{u}_i + \frac{h_1}{2}\,\boldsymbol{\nabla}\langle u_1^2\rangle,$$

$$\int\limits_{-h_2}^{\eta} w_2\,\frac{\partial \boldsymbol{u}_2}{\partial z}\,dz = \frac{\partial\eta}{\partial t}\,\boldsymbol{u}_i - \int\limits_{-h_2}^{\eta} \boldsymbol{u}_2\,\frac{\partial w_2}{\partial z}\,dz = \frac{\partial\eta}{\partial t}\,\boldsymbol{u}_i + \int\limits_{-h_2}^{\eta} \boldsymbol{u}_2\cdot\boldsymbol{\nabla}\boldsymbol{u}_2\,dz = \frac{\partial\eta}{\partial t}\,\boldsymbol{u}_i + \frac{h_2}{2}\,\boldsymbol{\nabla}\langle u_2^2\rangle.$$

To find the quantities $\nu_l\rho_l\,\partial\boldsymbol{u}_l/\partial z$ in the vicinity of all boundaries, one has first to solve the problem in the first approximation, i.e., with allowance for first-order terms only.

Determination of Shear Stress at the Lid, at the Bottom, and between the Fluid Layers. As not only the amplitude of long waves and surface tension are small but also the viscosity is low, nonlinear, inertial, capillary, and dissipative terms in Eqs. (8) and (9) can be neglected. Then, the linearized laws of conservation of mass and horizontal momentum for each layer have the form

$$\frac{\partial \eta}{\partial t} + (-1)^l h_l \, \boldsymbol{\nabla} \cdot \langle \boldsymbol{u}_l \rangle = 0; \tag{10}$$

$$h_l \, \frac{\partial \langle \boldsymbol{u}_l \rangle}{\partial t} + h_l \, \boldsymbol{\nabla} \Big(g\eta + \frac{p_i}{\rho_l} \Big) = 0. \tag{11}$$

The corrections characterized by negligible orders of smallness are again omitted; hence, $p_{1i} = p_{2i} = p_i$. To eliminate the fluid velocities, we differentiate Eqs. (10) in time, apply the operator $\boldsymbol{\nabla}$ multiplied by $(-1)^l$ to Eqs. (11) in a scalar manner, and subtract the latter equations from the former ones. As a result, we obtain

$$\frac{\partial^2 \eta}{\partial t^2} - (-1)^l \Big(g h_l \nabla^2 \eta + \frac{h_l}{\rho_l} \, \nabla^2 p_i \Big) = 0.$$

The Laplacian of the pressure disturbance at the interface is found under the condition of identity of these two equations (for $l = 1$ and 2), which describe the same wave process:

$$\nabla^2 p_i = -\rho_1 \rho_2 g H \nabla^2 \eta / \chi, \qquad H = h_1 + h_2, \qquad \chi = \rho_1 h_2 + \rho_2 h_1.$$

From here, we obtain the formula for the pressure gradient on the interface

$$\boldsymbol{\nabla} p_i = -\rho_1 \rho_2 g H \boldsymbol{\nabla} \eta / \chi. \tag{12}$$

We set the integration constant to zero because the fluids are quiescent in the absence of disturbances in accordance with the assumption made previously.

Thus, the linearized equations of motion (7) for very long waves at the interface (surface tension being neglected) become

$$\frac{\partial \boldsymbol{u}_l}{\partial t} + g \Big(1 - \frac{\rho_1 \rho_2 H}{\rho_l \chi} \Big) \boldsymbol{\nabla} \eta = \nu_l \, \frac{\partial^2 \boldsymbol{u}_l}{\partial z^2}.$$

We rewrite these equations in a form more convenient for further presentation:

$$\frac{\partial^2 \boldsymbol{u}_l}{\partial z^2} - \frac{1}{\nu_l} \, \frac{\partial \boldsymbol{u}_l}{\partial t} = (-1)^l \, \frac{c_0^2}{\nu_l h_l} \, \boldsymbol{\nabla} \eta, \qquad c_0^2 = \frac{g(\rho_2 - \rho_1) h_1 h_2}{\rho_1 h_2 + \rho_2 h_1}. \tag{13}$$

We seek for solutions of these differential equations by the method of separation of variables:

$$u_{lj}(t, x, y, z) = v_{lj}(t, z) f_{lj}(x, y), \qquad j = x, y.$$

Now we subject Eqs. (13) to the standard Laplace transform in time

$$\frac{\partial^2 V_{lj}}{\partial z^2} - \frac{s}{\nu_l} \, V_{lj} = \frac{G_{lj}}{\nu_l f_{lj}} - \frac{v_{lj,0}}{\nu_l} \equiv N_{lj}(s), \tag{14}$$

where the functions $V_{lj}(s, z)$ and $G_{lj}(s, x, y)$ are the images of $v_{lj}(t, z)$ and $(-1)^l (c_0^2 / h_l) \, \partial \eta / \partial j$, respectively. The right sides of Eqs. (14) are functions of the variable s only because their left sides are independent of the horizontal coordinates x and y and the fluid velocities in each layer at the initial time ($t = 0$) are independent of the vertical coordinate z. The assumptions of a small thickness of the boundary layers make it possible, without loss of accuracy, to impose a condition of the absence of shear stresses at large distances from the surfaces considered, i.e., for $z = -\infty$ (for the lid), for $z = +\infty$ (for the bottom), or for $z = \pm\infty$ simultaneously for the interface (the boundary layers are submerged into the fluid to an "infinite" depth).

The substitutions $V'_{lj} = V_{lj} + N_{lj} \, \nu_l / s$ allow us to rewrite Eqs. (14) in the form of homogeneous linear differential second-order equations. Then, we can easily find the solutions of interest, which satisfy the boundary conditions $V_{lj} = 0$ for $z = -(-1)^l h_l(x, y)$ and $\partial V_{lj} / \partial z = 0$ for $z = (-1)^l \infty$:

$$V_{lj}(s, z) = N_{lj}(s) \, \frac{\nu_l}{s} \Big[\exp \Big(- \sqrt{\frac{s}{\nu_l}} \, [(-1)^l z + h_l] \Big) - 1 \Big].$$

Therefore, we obtain the following relations for the derivatives of the images V_{lj} for the lid and bottom:

$$\frac{\partial V_{lj}}{\partial z}\Big|_{z=-(-1)^l h_l} = -(-1)^l N_{lj}(s)\sqrt{\frac{\nu_l}{s}}.$$

Applying the inverse Laplace transform to these formulas, we obtain expressions for tangential components of the shear stress tensor in the space of originals:

$$\tau_{l,jz} = \rho_l\sqrt{\frac{\nu_l}{\pi}}\Big((-1)^l\,\frac{u_{lj,0}}{\sqrt{t}} - \frac{c_0^2}{h_l}\int\limits_0^t \frac{\partial\eta}{\partial j}\,\frac{dt'}{\sqrt{t-t'}}\Big). \tag{15}$$

Similarly, by simultaneously solving Eqs. (14) with four boundary conditions [$\partial V_{lj}/\partial z = 0$ for $z = -(-1)^l\infty$, $V_{1j} = V_{2j}$ and $\nu_1\rho_1\,\partial V_{1j}/\partial z = \nu_2\rho_2\,\partial V_{2j}/\partial z$ for $z = 0$], we obtain the following profiles for the images V_{lj} in the vicinity of the interface:

$$V_{1j} = \sqrt{\nu_2}\rho_2\,\frac{\nu_1 N_{1j} - \nu_2 N_{2j}}{s(\sqrt{\nu_1}\rho_1 + \sqrt{\nu_2}\rho_2)}\exp\Big(-\sqrt{\frac{s}{\nu_1}}\,z\Big) - \frac{\nu_1}{s}\,N_{1j},$$

$$V_{2j} = \sqrt{\nu_1}\rho_1\,\frac{\nu_2 N_{2j} - \nu_1 N_{1j}}{s(\sqrt{\nu_1}\rho_1 + \sqrt{\nu_2}\rho_2)}\exp\Big(\sqrt{\frac{s}{\nu_2}}\,z\Big) - \frac{\nu_2}{s}\,N_{2j}.$$

Applying the inverse Laplace transform to these dependences, we find expressions for horizontal components of velocity and shear stress at the interface:

$$u_{ij} = \frac{c_0^2}{\psi_1+\psi_2}\Big(\frac{\psi_1}{h_1} - \frac{\psi_2}{h_2}\Big)\int\limits_0^t \frac{\partial\eta(t',x,y)}{\partial j}\,dt' + \frac{\psi_1 u_{1j,0}(x,y) + \psi_2 u_{2j,0}(x,y)}{\psi_1+\psi_2}; \tag{16}$$

$$\tau_{i,jz} = \nu_l\rho_l\,\frac{\partial u_{lj}}{\partial z}\Big|_{z=0} = \frac{\psi}{\sqrt{\pi t}}\,(u_{1j,0} - u_{2j,0}) + \frac{\psi}{\sqrt{\pi}}\,\frac{c_0^2 H}{h_1 h_2}\int\limits_0^t \frac{\partial\eta(t',x,y)}{\partial j}\,\frac{dt'}{\sqrt{t-t'}}. \tag{17}$$

Here $\psi_l = \sqrt{\nu_l}\rho_l$ and a dissipative coefficient $\psi = \psi_1\psi_2/(\psi_1+\psi_2)$ is introduced. Note that the time in formulas (15)–(17) is $t > 0$, and the nonintegral terms exert some effect only in the region disturbed at the initial time ($t = 0$). For the remaining space, the terms containing $u_{lj,0}(x,y)$ actually disappear (vanish).

Evolutionary Equation for Waves and Analysis of its Particular Solutions. We substitute relations (15) and (17) into Eqs. (9). Then, the equations of the balance of horizontal components of momentum for each layer of the fluid are written in the form

$$\frac{\partial([h_l + (-1)^l\eta]\langle\boldsymbol{u}_l\rangle)}{\partial t} + [h_l + (-1)^l\eta]\boldsymbol{\nabla}\Big(g\eta + \frac{p_{li}}{\rho_l}\Big) + h_l\boldsymbol{\nabla}\Big(\langle u_l\rangle^2 + (-1)^l\frac{h_l}{3}\frac{\partial^2\eta}{\partial t^2}\Big)$$

$$= \frac{(-1)^l}{\sqrt{\pi}\rho_l}\Big[\frac{\psi}{\sqrt{t}}\,(\boldsymbol{u}_{1,0} - \boldsymbol{u}_{2,0}) - \frac{(-1)^l\psi_l}{\sqrt{t}}\,\boldsymbol{u}_{l,0} + c_0^2\Big(\frac{\psi H}{h_1 h_2} + \frac{\psi_l}{h_l}\Big)\int\limits_0^t \boldsymbol{\nabla}\eta\,\frac{dt'}{\sqrt{t-t'}}\Big]. \tag{18}$$

To eliminate the averaged velocities of the fluids from linear terms, we differentiate Eqs. (8) with respect to time again, apply the operator $\boldsymbol{\nabla}$ to Eqs. (18) in a scalar manner, and subtract the latter equations multiplied by $(-1)^l$ from the former equations. As a result, we obtain

$$\frac{\partial^2\eta}{\partial t^2} - (-1)^l\Big(gh_l\,\nabla^2\eta + \frac{h_l}{\rho_l}\,\nabla^2 p_{li}\Big) - \frac{h_l^2}{3}\,\nabla^2\,\frac{\partial^2\eta}{\partial t^2} - (-1)^l\boldsymbol{\nabla}h_l\cdot\boldsymbol{\nabla}\Big(g\eta + \frac{p_{li}}{\rho_l}\Big)$$

$$-\boldsymbol{\nabla}\cdot\Big[\eta\,\boldsymbol{\nabla}\Big(g\eta + \frac{p_{li}}{\rho_l}\Big)\Big] - (-1)^l h_l\nabla^2\langle u_l^2\rangle + \sqrt{\frac{\nu_l}{\pi}}\,\frac{c_0^2}{h_l}\Big(1 + \frac{\psi\,h_l H}{\psi_l h_1 h_2}\Big)\int\limits_0^t \frac{\nabla^2\eta\,dt'}{\sqrt{t-t'}}$$

$$= \sqrt{\frac{\nu_l}{\pi t}}\,\boldsymbol{\nabla}\cdot\Big(\frac{\psi}{\psi_l}\,(\boldsymbol{u}_{2,0} - \boldsymbol{u}_{1,0}) + (-1)^l\boldsymbol{u}_{l,0}\Big).$$

Now the fluid velocities are found only in terms of the second order of smallness. Moreover, we confine our consideration to processes in which nonlinear waves travel only in a certain direction. Then we can replace $(-1)^l h_l(\boldsymbol{\nabla}\cdot\boldsymbol{u}_{l,0})$ by $(\mathbf{c}_0\cdot\boldsymbol{\nabla}\eta_0)$ in the right sides of these equations and $\langle u_l^2\rangle$ by $c_0^2\eta^2/h_l^2$ in the fifth terms. Finally, in second-order terms, we can replace pressure gradients at the interface of the fluid layers with the use of Eqs. (12). As a result, we obtain the following system of equations:

$$\frac{\partial^2\eta}{\partial t^2}-(-1)^l\Big(gh_l\nabla^2\eta+\frac{h_l}{\rho_l}\,\nabla^2 p_{li}\Big)-(-1)^l\,\frac{3}{2}\,\frac{c_0^2}{h_l}\,\nabla^2\eta^2-\frac{h_l^2}{3}\,\nabla^2\,\frac{\partial^2\eta}{\partial t^2}-\frac{c_0^2}{h_l}\,(\boldsymbol{\nabla}h_l\cdot\boldsymbol{\nabla}\eta)$$

$$=\sqrt{\frac{\nu_l}{\pi}}\,\frac{1}{h_l}\Big(1+\frac{\psi\,h_l H}{\psi_l h_1 h_2}\Big)\Big(\frac{\boldsymbol{c}_0}{\sqrt{t}}\cdot\boldsymbol{\nabla}\eta_0-c_0^2\int\limits_0^t\frac{\nabla^2\eta\,dt'}{\sqrt{t-t'}}\Big). \tag{19}$$

To reduce the system of these two equations to one equation (eliminating the pressures at the interface from the linear terms as well, i.e., from terms of the first order of smallness), we multiply Eq. (19) by h_2/ρ_2 for $l=1$ and by h_1/ρ_1 for $l=2$ and add them to obtain

$$\frac{\partial^2\eta}{\partial t^2}-\frac{g(\rho_2-\rho_1)h_1h_2}{\chi}\,\nabla^2\eta-\frac{3}{2}\,\frac{\rho_2h_1^2-\rho_1h_2^2}{h_1h_2\chi}\,c_0^2\nabla^2\eta^2-\frac{h_1h_2}{3}\,\frac{\rho_1h_1+\rho_2h_2}{\chi}\,\nabla^2\frac{\partial^2\eta}{\partial t^2}$$

$$+\frac{\sigma\,h_1h_2}{\chi}\,\nabla^4\eta-\frac{\rho_1h_2c_0^2}{h_1\chi}\,\boldsymbol{\nabla}\eta\cdot\boldsymbol{\nabla}h_1-\frac{\rho_2h_1c_0^2}{h_2\chi}\,\boldsymbol{\nabla}\eta\cdot\boldsymbol{\nabla}h_2$$

$$=\frac{\psi H^2+\psi_1h_2^2+\psi_2h_1^2}{\sqrt{\pi}h_1h_2\chi}\Big(\frac{\boldsymbol{c}_0}{\sqrt{t}}\cdot\boldsymbol{\nabla}\eta_0-c_0^2\int\limits_0^t\frac{\nabla^2\eta\,dt'}{\sqrt{t-t'}}\Big), \tag{20}$$

where $\eta_0(x,y)$ is the initial deviation of the free surface. Thus, the coefficients at all terms in Eq. (20) are determined only by geometrical (h_1 and h_2) and physical ($g,\sigma,\rho_1,\rho_2,\nu_1$, and ν_2) parameters of the problem.

Within the assumptions made above (low frequency of disturbances, moderate amplitude of disturbances, their propagation in one direction only, gentle slope of the solid surfaces, and small thickness of boundary layers), we can replace $\partial^2/\partial t^2$ by $c_0^2\nabla^2$ in terms of the second order of smallness and vice versa. Then, Eq. (20) becomes

$$\frac{\partial^2\eta}{\partial t^2}-c_0^2\,\nabla^2\eta-C_n\,\nabla^2\eta^2-C_d\,\nabla^2\frac{\partial^2\eta}{\partial t^2}-C_{b1}\,\boldsymbol{\nabla}\eta\cdot\boldsymbol{\nabla}h_1-C_{b2}\,\boldsymbol{\nabla}\eta\cdot\boldsymbol{\nabla}h_2=\frac{C_f}{\sqrt{\pi}}\Big(\frac{\boldsymbol{c}_0}{\sqrt{t}}\cdot\boldsymbol{\nabla}\eta_0-c_0^2\int\limits_0^t\frac{\nabla^2\eta\,dt'}{\sqrt{t-t'}}\Big). \tag{21}$$

The evolutionary equations (20) and (21) take into account weak nonlinearity of disturbances, long-wave contributions of inertia of the fluid layers and surface tension, gentle slopes of the lid and bottom, and nonstationary shear stresses at all boundaries of this system.

We emphasize that Eqs. (20) and (21) can be used to describe transformations of essentially three-dimensional disturbances moving in an arbitrary horizontal direction (at an arbitrary angle to the Ox axis). In this case, linear waves can simultaneously travel in the opposite directions.

It is well known (see, e.g., monograph [2]) that various effects of the transition to finite differences and rounding in numerical calculations lead to small high-frequency oscillations even if the analytical problem under study satisfies the long-wave condition. For this reason, numerical solutions found with the use of Eq. (21) can turn to be unstable for $C_d<0$ (short waves are amplified). In such a situation, one has to return to the former type of the dispersion term, i.e., to Eq. (20).

Let us analyze some particular and limiting cases. First we consider an ideal fluid. In this case, Eqs. (20) and (21) are essentially simplified. The fact that they become differential is only one aspect. An important point is that a convolution rather than the mere integral disappears, because calculating the convolution requires the knowledge of the history of the process in the entire channel under study, i.e., tremendous computer memory and high performance. Let $\eta(t,x,y)=\eta(t,x)$ and $dh_l/dx=0$; then Eq. (21) agrees with the modified Boussinesq equation [9]; for disturbances propagating in the direction of the increasing x coordinate, we obtain a traditional evolutionary equation of the Korteweg–de Vries type. These equations are similar to that derived in [13]. Therefore, the dispersion analysis performed there is applicable to the system under study as well. In addition, we can easily find that the stationary two-dimensional periodic solutions of Eq. (21) with $\nu_l = 0$ and $dh_l/dx = 0$ are the cnoidal waves

$$\eta = \eta_a \operatorname{cosn}^2 [(x - U_{\rm cn} t)/L_{\rm cn}].$$

Here $U_{\rm cn} = c_0\sqrt{1 + \eta_a^*(2 - 1/m^2)}$, $\eta_a^* = 2\eta_a C_n/(3c_0^2)$, and $L_{\rm cn} = m\, U_{\rm cn}\sqrt{6C_d/(\eta_a C_n)}$. The parameter m (absolute value of the elliptic Jacobian function) determines the level of nonlinearity of the disturbance (see, e.g., monograph [2]). Thus, as $m \to 0$, we obtain a harmonic wave; as $m \to 1$, the length of disturbances tends to infinity, and we obtain solitary solutions (see [9])

$$\eta = \eta_a / \cosh^2 [(x - Ut)/L], \tag{22}$$

where $U = c_0\sqrt{1 + \eta_a^*}$ and $L = L_0 = 2\sqrt{C_d(1 + 1/\eta_a^*)}$. If $\partial^2\eta/\partial y^2 \sim \varepsilon\, \partial^2\eta/\partial x^2$ and $\boldsymbol{\nabla} h_l = 0$, Eq. (21) is simplified to an extent that it can be reduced to an equation of the Kadomtsev–Petviashvili type, which is also well known [14]. Thus, Eq. (21) is an extension of a quasi-two-dimensional evolutionary equation to the case of essentially three-dimensional disturbances even for $\nu_l = 0$ and $\boldsymbol{\nabla} h_l = 0$.

Pelinovsky and Stepanyants [15] applied a two-dimensional equation of the type (21) with $\nu_l = 0$ and $\boldsymbol{\nabla} h_l = 0$ to study the streamwise and crossflow instability of solitary waves in homogeneous media with positive dispersion (i.e., in our situation, with $C_d < 0$). In particular, spatial solitons and solitary disturbances with a periodically modulated front were considered.

Concerning the slopes of the lid and bottom, allowance for these factors with the help of only two terms in Eq. (21) taking into account deviations of the boundaries from the horizontal direction is sufficient to obtain the Green law: with decreasing depth of the fluid, the amplitude of a two-dimensional linear wave is inversely proportional to fourth-power roots from the depth of the layer.

Now let us take into account the fluid viscosity. The dispersion analysis of this case is also similar to that performed in [13]. Calculating the convolution in Eq. (21) requires the entire history of the process to be known. Yet, the values of $\partial\eta/\partial j$ ($j = x, y$) that refer to the time $t' \ll t$ are added with a smaller "weight" than the values of $\partial\eta/\partial j$ at $t' \approx t$. Therefore, the values of shear stress at the lid and bottom (the latter can be horizontal) change their sign owing to propagation of even two-dimensional solitary disturbances [13]. This effect occurs because the streamwise components of pressure gradients at the rear front of the wave act against disturbances of fluid velocities. Hence, separations of thin boundary layers and origination of backflow zones are possible [13]. Unfortunately, the author is not aware of any experimental investigations of this phenomenon.

Numerical Solutions of the Model Evolutionary Equation. As Eq. (21) is similar to the equation for waves on the free surface of a homogeneous fluid, the calculations were performed with the use of an implicit three-layer finite-difference scheme described in [16]. This scheme has the second order of approximation in terms of all three variables and possesses good stability.

Below we consider some problems with a two-dimensional steady solitary wave (22) used as an initial disturbance (in the first and second time layers). Therefore, the partial derivative with respect to time at the initial moment was defined obviously. Reflection conditions were set on the vertical side walls of the channel, which implied that the derivatives of η in directions normal to the walls were set to zero. For this purpose, the values of the sought function η in boundary nodes were equated to its values in the nearest inner node.

This difference scheme was tested by analytically solving Eq. (21) with $\nu_l = 0$ and $\boldsymbol{\nabla} h_l = 0$ in the form of a two-dimensional solitary disturbance (22). The computations were validated not only to be stable but also to converge with at least the second order for both variables (both the spatial coordinate and time). Figures 2–5 show the results computed for a system of two fluids: kerosene and water (densities $\rho_1 = 800$ kg/m^3 and $\rho_2 = 1000$ kg/m^3, kinematic viscosities $\nu_1 = 1.62$ mm^2/sec and $\nu_2 = 1.08$ mm^2/sec, and surface tension $\sigma = 34$ mN/m).

If the initial wave is moderately long [Eq. (22) with $L = L_0/2$], the leading front of the disturbance becomes less steep, and oscillations with a gradually decreasing amplitude arise behind this front (Fig. 2). The initial wave can be said to decompose into one soliton and a wave packet (see, e.g., [1]). In such a situation, taking into account of nonstationary shear stresses at all boundaries of the system (solid curves) leads only to decay of disturbances (the dashed curves refer to ideal fluids). The undisturbed depths of the fluid layers are $h_1 = 12$ cm and $h_2 = 8$ cm, the step over the coordinate is $\Delta x = 2$ cm, and the time step is $\Delta t = 0.125$ sec.

If the initial wave is very long [Eq. (22) with $L = 4L_0$], the disturbance certainly transforms first to a "triangular" disturbance with a steep leading front and extended rear front, and then it can be expected to yield a

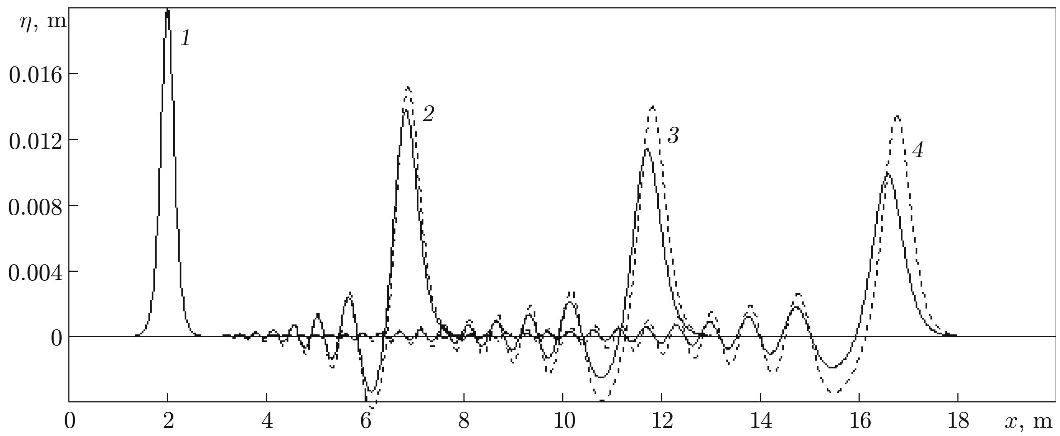

Fig. 2. Transformation of a moderately long nonlinear solitary wave in a channel with a horizontal lid and bottom: $t = 0$ (1), 15 (2), 30 (3), and 45 sec (4).

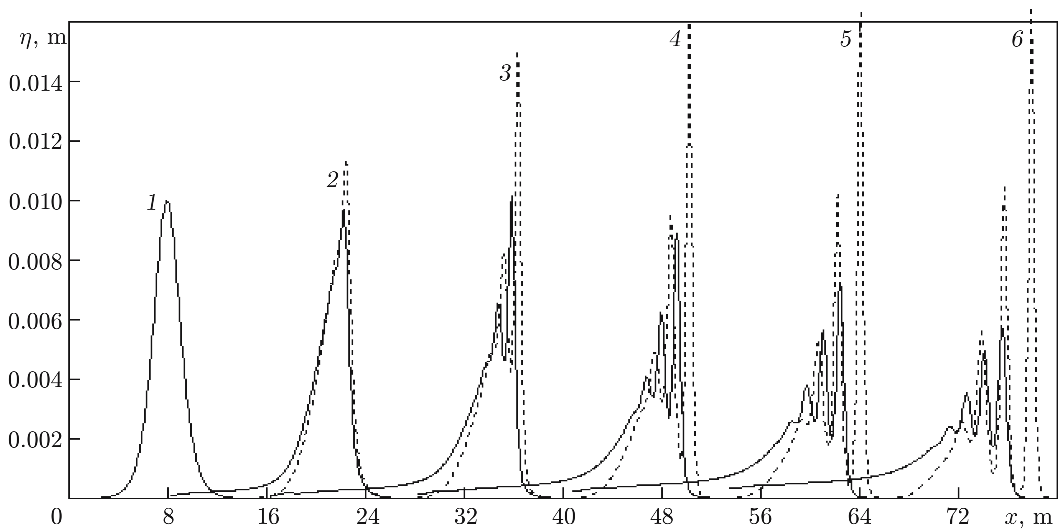

Fig. 3. Propagation of a very long solitary disturbance in a system with a horizontal lid and bottom: $t = 0$ (1), 40 (2), 80 (3), 120 (4), 160 (5), and 200 sec (6).

chain of solitary waves with decreasing amplitude (see Fig. 3). Nevertheless, the presence of viscosity prevents this process. Here we again have $h_1 = 12$ cm and $h_2 = 8$ cm, but $\Delta x = 8$ cm and $\Delta t = 0.25$ sec; the notation of curves is the same as that in Fig. 2.

Let us now consider the evolution of a long solitary disturbance in a channel with an inclined segment of the lid. Figure 4 shows the results computed for the case where the depth of the upper fluid is constant at the initial segment 8 m long and then linearly decreases by 25% (from 12 to 9 cm) over the segment from 8 to 16 m. The initial wave is described by profile (22) with $L = L_0$, $\Delta x = 4$ cm, and $\Delta t = 0.125$ sec. As compared to the situation of a horizontal lid, the main disturbance is now followed by a small disperse "tail" in the form of a wave packet (dashed curves). Allowance of nonstationary shear stresses at all boundaries of the system (solid curves) leads to even earlier decay of the basic wave and origination of the dissipative "tail." This results not only in decay of oscillations but also in the absence of negative values of the interface disturbance.

Figure 5 shows the results computed for the case where the depth of the upper layer decreases linearly by 50% (from 12 to 6 cm) over the segment from 8 to 16 m. The initial wave is again described by profile (22) with $L = L_0$, $\Delta x = 4$ cm, and $\Delta t = 0.125$ sec. Not only an obvious decrease in disturbance velocity but also a drastic change in the disturbance shape are clearly visible. As in the case shown in Fig. 2, origination of the wave packet

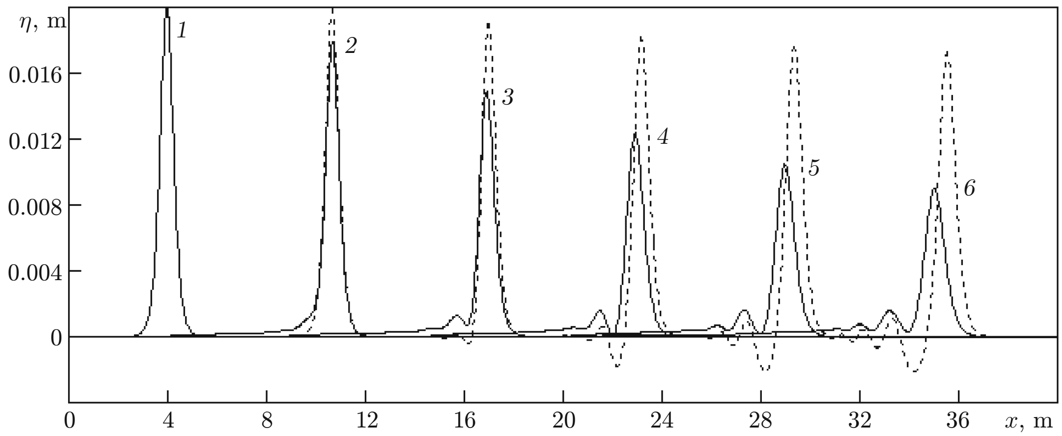

Fig. 4. Evolution of a nonlinear solitary disturbance in a system with a slightly downward lid and a horizontal bottom: $t = 0$ (1), 20 (2), 40 (3), 60 (4), 80 (5), and 100 sec (6).

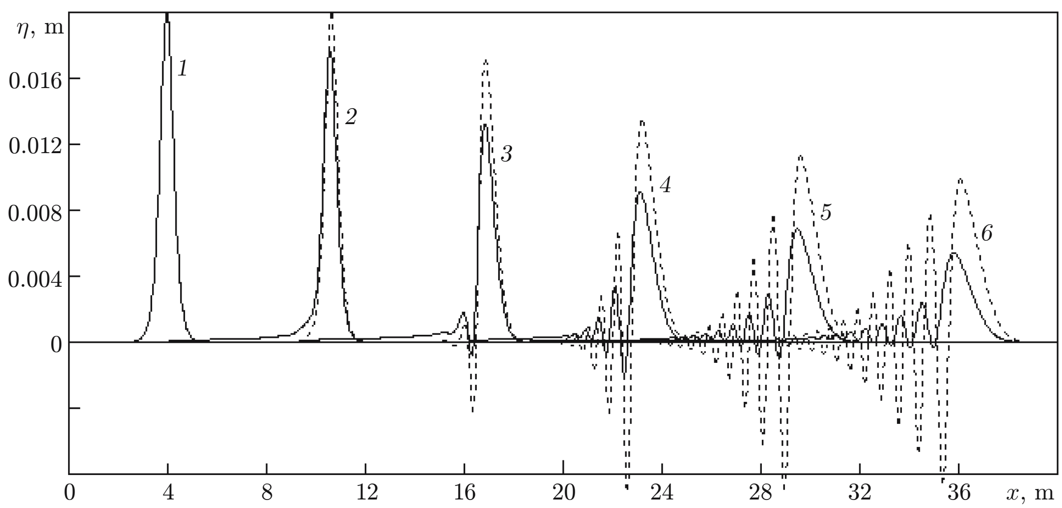

Fig. 5. Transformation of a nonlinear solitary wave in a system with a significant downward slope of the lid and a horizontal bottom: $t = 0$ (1), 20 (2), 42 (3), 60 (4), 90 (5), and 114 sec (6).

is observed, and the leading front becomes less steep. As the lid become lower, the coefficient at the nonlinear term in Eq. (21) changes its sign, which is responsible for this manner of disintegration of the initial wave. The effect of viscosities of the fluids is better seen in this situation.

Equation (21) allowed us to calculate the transformation of an initially two-dimensional nonlinear solitary wave (22) in a channel with a three-dimensional obstacle. Figure 6 shows the interface disturbances for the case where the bottom is horizontal and the lid has an almost smooth prominence aligned with the wave vector:

$$\begin{aligned} &h_1\{1 - 0.5\exp[-4(x_r - x)^2]\exp[-(y_r - y)^2]\} \qquad &&\text{for} \quad x \le x_r, \\ &h_1\{1 - 0.5\exp[-(y_r - y)^2]\} &&\text{for} \quad x > x_r \end{aligned}$$

(x_r and y_r are the characteristic coordinates of the obstacle). In the present situation, $x_r = 4$ m and $y_r = 3$ m. The remaining parameters of the fluids are the same as those in the computations described above (see Figs. 2–5). A certain lag of disturbances and weak ship-type waves are observed. Nevertheless, Fig. 6 allows one to see not a mere increase in disturbance amplitude under the prominence and formation of the dispersion "tail" but also the disintegration of the solitary wave in this part of the channel, though decay due to dissipation made this process noticeably smoother.

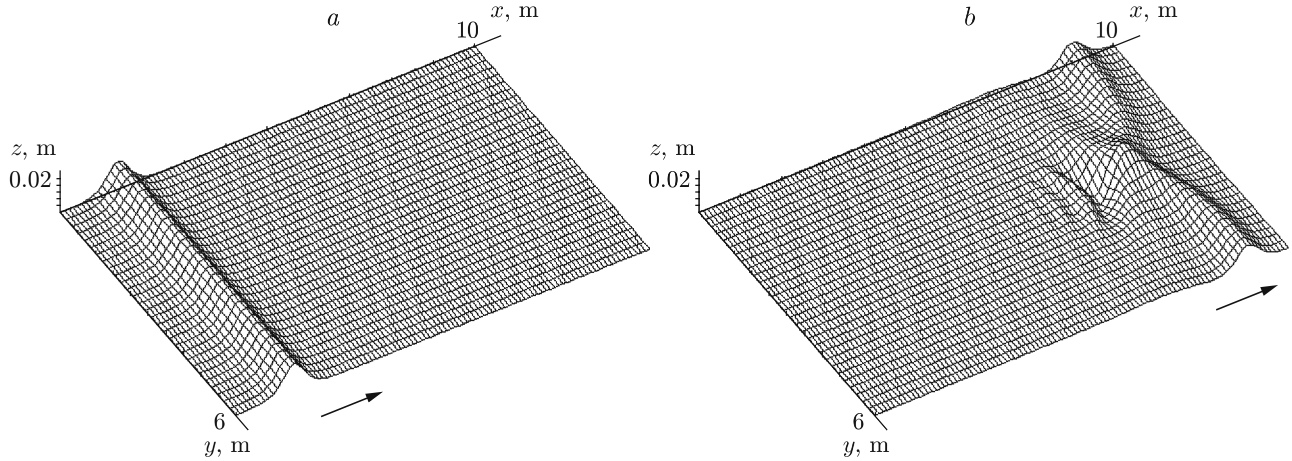

Fig. 6. Propagation of a two-dimensional nonlinear solitary wave in a channel with a three-dimensional prominence on the lid: $t = 0$ (a) and $t = 23$ sec (b).

Discussion of Computation Results. The evolution of long solitary disturbances in stratified fluids with a horizontal bottom, similar to that considered above, was observed by many authors by using the Korteweg–de Vries equation (see, e.g., monographs [1, 4]) and in various experiments (see [4, 7, 17]).

The results calculated for an internal solitary wave in a two-layer fluid under a lid with smooth uprising of the bottom are described in [18]. The disturbance dynamics shown in Fig. 5 is in good agreement with these results. The experimental study [19] of the transformation of an internal solitary wave in a two-layer fluid with a free surface was performed in a tray with a segment of a linear decrease in the depth of the lower fluid. The qualitative pattern is the same as that in Fig. 5, and a quantitative comparison cannot be made because the marigrams in the illustrations in [19] are too small.

A similar situation was observed in full-scale observations and in model computations for nonlinear periodic waves [20]. In [21], the coefficient at the nonlinear term in the Korteweg–de Vries equation also changed its sign. This was attributed, however, to the horizontal inhomogeneity of the depths of the fluid layers, i.e., to the interface position, rather than to bottom uprising.

Concerning the evolution of essentially three-dimensional disturbances simulated by Eq. (21), it is similar to that considered in [16] for rather long waves of small but finite amplitude in a homogeneous layer of a viscous fluid above a gently sloping bottom with a correction to possible "upturned" disturbances with appropriate parameters of the two-layer system and lid irregularities.

Conclusions. An evolutionary equation is derived for three-dimensional waves at the interface of two viscous fluids, which takes into account weak nonlinearity of disturbances, long-wave contributions of inertia of the layers and surface tension, gentle slopes of the lid and bottom, and nonstationary shear stresses at all boundaries of the system. The numerical solutions of the model equation for two-dimensional waves are in good agreement with results obtained by other authors. A possible contribution of dissipation to dynamics of two-dimensional solitary disturbances of small but finite amplitude is demonstrated. Transformation of an initially two-dimensional nonlinear wave to a three-dimensional disturbance, which is induced by a smooth prominence on the lid, is calculated. Thus, the equation obtained substantially simplifies the study of wave processes in channels of complicated configurations for three-dimensional disturbances as well.

REFERENCES

1. V. I. Karpman, *Nonlinear Waves in Dispersive Media*, Pergamon Press, New York (1975).
2. G. B. Whitham, *Linear and Nonlinear Waves*, Wiley Interscience, New York (1974).
3. L. V. Ovsyannikov, N. I. Makarenko, V. I. Nalimov, et al., *Nonlinear Problems of the Theory of Surface and Internal Waves* [in Russian], Nauka, Novosibirsk (1985).

4. M. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Theory*, SIAM, Philadelphia (1981).
5. V. Yu. Liapidevskii and V. M. Teshukov, *Mathematical Models of Propagation of Long Waves in an Inhomogeneous Fluid* [in Russian], Izd. Sib. Otd. Ross. Akad. Nauk, Novosibirsk (2000).
6. A. T. Il'ichev, *Solitary Waves in Hydrodynamic Models* [in Russian], Fizmatlit, Moscow (2003).
7. C. G. Koop and G. Butler, "An investigation of internal solitary waves in a two-fluid system," *J. Fluid Mech.*, **112**, 225–251 (1981).
8. C. Leone, H. Segur, and J. L. Hammack, "Viscous decay of long internal solitary waves," *Phys. Fluids*, **25**, No. 6, 942–944 (1982).
9. G. A. Khabakhpashev, "Evolution of disturbances of the interface between two layers of viscous fluid," *Fluid Dynamics*, **25**, No. 6, 909–913 (1990).
10. G. H. Keulegan, "Gradual damping of solitary waves," *J. Res. Nat. Bur. Stand.*, **40**, No. 6, 487–498 (1948).
11. T. Kakutani and K. Matsuuchi, "Effect of viscosity on long gravity waves," *J. Phys. Soc. Japan*, **39**, No. 1, 237–246 (1975).
12. N. V. Gavrilov, "Viscous attenuation of solitary internal waves in a two-layer fluid," *J. Appl. Mech. Tech. Phys.*, **29**, No. 4, 503–506 (1988).
13. G. A. Khabakhpashev, "Modeling the propagation of internal waves in a two-layer ocean," *Izv., Atmospher. Ocean. Phys.*, **26**, No. 1, 47–54 (1990).
14. B. B. Kadomtsev and V. I. Petviashvili, "Steadiness of solitary waves in weakly dispersive media," *Soviet Phys. Dokl.*, **15**, 539–542 (1970).
15. D. E. Pelinovsky and Yu. A. Stepanyants, "Unsteadiness of solitary waves in media with positive dispersion in the framework of two-dimensional Boussinesq equations," *J. Exper. Theor. Phys.*, **79**, No. 1 (1994).
16. A. A. Litvinenko and G. A. Khabakhpashev, "Numerical simulation of sufficiently long nonlinear two-dimensional waves on water in pools with a gently sloping bottom," *Vychisl. Tekhnol.*, **4**, No. 3, 95–105 (1999).
17. H. Segur and J. L. Hammack, "Solitary models of long internal waves," *J. Fluid Mech.*, **118**, 285–304 (1982).
18. K. R. Helfrich, W. K. Melville, and J. W. Miles, "On interfacial solitary waves over slowly varying topography," *J. Fluid Mech.*, **149**, 305–317 (1984).
19. J. Maurer, K. Hutter, and S. Diebels, "Viscous effect in internal waves of two-layered fluid with variable depth," *Europ. J. Mech., B: Fluids*, **15**, No. 4, 445–470 (1996).
20. P. E. Holloway, E. Pelinovsky, T. Talipova, and B. Barnes, "A nonlinear model of internal tide transformation on the Australian North West Shelf," *J. Phys. Oceanogr.*, **27**, No. 6, 871–896 (1997).
21. T. G. Talipova, E. N. Pelinovsky, and R. Grimshaw, "Transformation of soliton in the point of zero nonlinearity," *JETP Lett.*, **65**, No. 1, 113–117 (1997).